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The Rev. Professor Graves communicated the following method of solving a large class of linear differential equations by the application of certain theorems in the calculus of operations:—

1. If ϕ and ψ be any functions whatsoever of x, and m and r any numbers, positive or negative, whole or fractional, the symbolic equation

$$\left(D+\phi+\frac{m\psi'}{\psi}\right)\psi^r=\psi^r\left(D+\phi+\frac{(m+r)\psi'}{\psi}\right)$$

holds good for any subject which we may conceive operated on by its two members.

It will be convenient to put

$$D+\phi+\frac{m\psi'}{\psi}=A_m,$$

so that the preceding equation may be written in the form

$$A_m\psi^r=\psi^r A_{m+r}.$$

And operating on this again with the symbol $\psi^{-r}(\quad)\psi^{-r}$, we get

$$\psi^{-r}A_m=A_{m+r}\psi^{-r}.$$

2. It is easy to show that, for λ and μ any functions of x,

$$(D+\lambda)(D+\mu)-(D+\mu)(D+\lambda)=\mu'-\lambda'.$$

Therefore, if χ be any function of x, and m any number,

$$A_m(D+\chi)=(D+\chi)A_m+\chi'-\phi'-m\left(\frac{\psi'}{\psi}\right)',$$

whence

$$A_rA_0(D+\chi)=\left\{(D+\chi)A_r+\chi'-\phi'-r\left(\frac{\psi'}{\psi}\right)'\right\}A_0+A_r(\chi'-\phi').$$

If we now suppose that

$$\chi'-\phi'=c\psi^{-r}, \tag{1}$$

where c is some constant, this becomes

$$A_rA_0(D+\chi)=(D+\chi)A_rA_0+\left\{2c\psi^{-r}-r\left(\frac{\psi'}{\psi}\right)'\right\}A_0,$$

whence again,

$$A_{2r}A_rA_0(D+\chi) = \left\{(D+\chi)A_{2r} + c\psi^{-r} - 2r\left(\frac{\psi'}{\psi}\right)'\right\}A_rA_0$$
$$+A_{2r}\left\{2c\psi^{-r} - r\left(\frac{\psi'}{\psi}\right)'\right\}A_0.$$

But if we further suppose that

$$\left(\frac{\psi'}{\psi}\right)' = k\psi^{-r}; \qquad (2)$$

k being some constant, the last equation assumes the simpler form,

$$A_{2r}A_rA_0(D+\chi) = (D+\chi)A_{2r}A_rA_0 + (3c-3rk)\psi^{-r}A_rA_0.$$

And continuing the same process, we should find generally

$$A_{nr}A_{(n-1)r}\ldots A_rA_0(D+\chi) = (D+\chi)A_{nr}A_{(n-1)r}\ldots A_rA_0$$
$$+\left\{(n+1)c - \frac{n(n+1)}{2}rk\right\}\psi^{-r}A_{(n-1)r}\ldots A_rA_0.$$

Or, since by the theorem in § 1 the variable part of the last term is equal to

$$A_{nr}A_{(n-1)r}\ldots A_{2r}A_r\psi^{-r},$$

$$A_{nr}A_{(n-1)r}\ldots A_r\left[A_0(D+\chi) - \left\{(n+1)c - \frac{n(n+1)}{2}rk\right\}\psi^{-r}\right]$$
$$= (D+\chi)A_{nr}A_{(n-1)r}\ldots A_rA_0.$$

3. This last formula enables us to effect the solution of the linear differential equation

$$\left[(D+\phi)(D+\chi) - \left\{(n+1)c - \frac{n(n+1)}{2}rk\right\}\psi^{-r}\right]y = X, \qquad (3)$$

whenever the conditions (1) and (2) are satisfied; as it furnishes us in that case with the means of inverting the operator in the left-hand member. Thus we find

$$y = A_0^{-1}A_r^{-1}\ldots A_{nr}^{-1}(D+\chi)^{-1}A_{nr}\ldots A_{2r}A_rX.$$

4. As regards the conditions (1) and (2), it will be observed, that the latter limits the *nature* of the function ψ, whilst the former makes the difference between ϕ and χ to depend upon that same function.

The solution of the equation

$$\left(\frac{\psi'}{\psi}\right)' = k\psi^{-r}$$

may be obtained by putting $\psi = z^s$ where s is as yet indeterminate; thus it is reduced to

$$s\left(\frac{z'}{z}\right)' = kz^{-rs}.$$

And if we now determine s by making $rs = 2$, we get

$$zz'' - z'^2 = \frac{rk}{2}.$$

Differentiating this again, we find

$$zz''' - z'z'' = 0,$$

the integral of which is

$$z'' \pm a^2 z = 0;$$

therefore,

$$z = (D^2 \pm a^2)^{-1}\, 0,$$

and

$$\psi = \{(D^2 \pm a^2)^{-1}\, 0\}^{\frac{2}{r}}.$$

5. To exemplify this theory we may assume $\psi = x^m$;

whence $\left(\dfrac{\psi'}{\psi}\right)' = -\dfrac{m}{x^2}$, $r = \dfrac{2}{m}$, $k = -m$, and $\theta - \phi = c_1 - \dfrac{c}{x}$.

The general formula becomes, therefore, in this case

$$\left\{(D + \phi)\left(D + \phi + c_1 - \frac{c}{x}\right) - \frac{(n+1)(c+n)}{x^2}\right\} y = X.$$

By making $c = 0$, and writing $-m$ in place of n, this becomes

$$\left\{(D + \phi)(D + \phi + c_1) - \frac{m(m-1)}{x^2}\right\} y = X,$$

which is equivalent to a general soluble form which Dr. Hargreave has obtained by an entirely different method.*

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* Phil. Trans., 1848, p. 35.

6. Or we may assume $\psi = (\cos x)^{2m}$; whence

$$\left(\frac{\psi'}{\psi}\right) = -2m \sec^2 x, \quad r = \frac{1}{m}, \quad k = -2m, \text{ and } \theta - \phi = c_1 + c \tan x.$$

In this case the general formula (3) becomes

$$[(D+\phi)(D+\phi+c_1+c\tan x)-(n+1)(c+n)\sec^2 x]\,y = X.$$

By putting $\phi = c_2$, and $c = 0$, this is reduced to

$$\{(D+c_1)(D+c_2) - n(n+1)\sec^2 x\}\,y = X,$$

which is the equation solved by Dr. Hargreave at p. 52, in the Paper already referred to, and from a particular case of which he derives the solution of the equation of Laplace's functions.

Sir Robert Kane read a paper by the Rev. Professor Callan, on the results of a series of experiments on the decomposition of water by the iron galvanic battery, with the view of obtaining a brilliant lime light.